Solvable statistical models on a random lattice

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We give a sequence of equivalent formulations of the ADE and $\hat{A}\hat{D}\hat{E}$ height models defined on a random triangulated surface: random surfaces immersed in Dynkin diagrams, chains of coupled random matrices, Coulomb gases, and multicomponent Bose and Fermi systems representing soliton τ-functions. We also formulate a set of loop-space Feynman rules allowing to calculate easily the partition function on a random surface with arbitrary topology. The formalism allows to describe the critical phenomena on a random surface in a unified fashion and gives a new meaning to the ADE classification.

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1. INTRODUCTION

During the past decade, stunning progress has been made in understanding and solving of statistical models on two-dimensional *random* lattices. The interest in these systems arose mainly because of their interpretation as models of discretized quantum gravity or, equivalently, "non-critical" string theories. The geometry of the lattice is considered here as an additional fluctuating variable.

The simplest statistical model on a random lattice is the random lattice itself (the target space consists of a single point). This problem goes back to the problem of counting planar diagrams solved in the seminal paper [1] after having been reformulated in terms of a large N matrix integral. More recently, some nontrivial statistical models have been solved on a random lattice following the same principle: the Ising model [2], the $O(n)$ model [3], the Q-state Potts model [4], and the ADE height models [5]. Each such system is equivalent to an ensemble of coupled random matrices labeled by the points of the target space of the model. Besides the models listed above, there are many other solvable ensembles of matrix models whose geometrical meaning still remains to be understood.

The statistical systems on random lattice are interesting not only as toy models of discretized quantum gravity. In spite of their simplicity, the models on a random lattice still contain considerable information about the critical behavior of the models on a regular lattice. Therefore they can be used as an heuristic instrument for studying complicated situations on regular lattices. For example, the dilute critical regime of the ADE models has been first discovered first on a random [6] and then on a regular lattice [7]. Furthermore, there are infinitely many multicritical regimes known on a random lattice, which are not yet considered on the regular lattice. We believe that the correspondence between the models on random and regular lattice merits to be understood on a deeper level and might help to establish the missing connections between the various resolution methods used by now.

In this talk I would like to review the construction and the solution of the ADE and $\hat{A}\hat{D}\hat{E}$ height models on a random lattice, as well as to present several equivalent formulations of these models as simple integrable systems.

It is well known [8] that the minimal two-dimensional conformal-invariant QFT are classified by the simply laced Lie algebras (i.e., these of the classical series A_r, D_r, E_6, E_7, E_8). Each

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of these theories can be constructed microscopically as a lattice statistical model (a *height* model), whose local degrees of freedom (heights) are labeled by the points of the Dynkin diagram of the corresponding Lie algebra [9]. Similarly, the height models associated with the extended $\hat{A}\hat{D}\hat{E}$ Dynkin diagrams describe conformal invariant QFT with $C=1$ and discrete spectrum of conformal dimensions. The height models in their dense (dilute) version can be mapped onto the 6-(19-)vertex model and solved using Bethe Anzatz [10] ([11]).

The height models can be defined on an arbitrary surface made of triangles or squares in such a way that the mapping onto a vertex model still holds. In the next section we will remind the definition of the height models on an arbitrary triangulated surface. It requires supplementary Boltzmann factors associated with the points with nonzero local curvature on the surface. Then we describe briefly the loop gas representation and the loop diagram technique that follows.

The models on a random lattice become topological in the sense that their correlation functions do not depend any more on the distance. The only parameters left are the topology of the lattice, the volume and the length of the boundaries and the boundary conditions. The local scaling operators can be introduced by shrinking a boundary with given boundary condition.

In Sect. 3. the height models on random lattice are then reformulated as systems of coupled random matrices, or, in the eigenvalue representation, as Coulomb gas systems.

A complete set of observables in the height models on a random surface is given by the set of all loop amplitudes or, in terms of free bosons or fermions, the correlation functions of currents. A genus g, n-loop amplitude is equal to the partition function of the model on a random surface with the topology of a sphere with g handles and n boundaries with given lengths. In Sect. 4. we present the loop diagram technique established in [13] and allowing to compute any loop amplitude on a surface with any topology. It will be formulated as a set of Feynman rules involving propagators, vertices (including tadpoles) of all topologies, and leg factors for the external boundaries. A vertex of given topology factorizes into a fusion coefficient for the order parameters of the height model and an "intersection number" associated with the corresponding punctured surface. In Sect. 5. the partition functions are reformulated, using the vertex operator construction, as systems of free bosons or fermions (soliton τ-functions).

2. INTEGRABLE HEIGHT MODELS LABELED BY DYNKIN DIAGRAMS

2.1. The target space

Let X be the Dynkin graph a simply laced Lie algebra of rank r. Such a graph consists of a set of r nodes, labeled by an integer *height* $x \in \{1,2,...,r\}$, and a number of bonds between nodes. Two nodes are called adjacent ($\sim$) on X if they are connected by a bond. The graph X is defined by its adjacency matrix

$$A^{xx'} = \begin{cases} 1, & \text{if } x \sim x' \ ; \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

which is related to the Cartan matrix of the scalar products of the simple roots $\vec{\alpha}^1, ..., \vec{\alpha}^R$ by

$$\vec{\alpha}^x \cdot \vec{\alpha}^{x'} = 2\delta_{xx'} - A^{xx'}. \tag{2}$$

Each Dynkin graph X can be extended to a graph $\hat{X}$ with $r+1$ nodes by adding an extra node representing the lowest root. The graph $\hat{X}$ can be considered as the Dynkin graph of the corresponding affine Lie algebra. For example, the Dynkin diagram of the affine algebra $\hat{A}_r$ ($R=1,2,...$) represents a closed chain of $R+1$ points. One can consider as well the case $R=0$ where the target space, which will be denoted by $\hat{A}_0$, consists of one oriented link with identified ends. The connectivity matrix of this target space is $A_{00}=2$.

In what follows we assume that the target space X is a Dynkin graph of an extended Dynkin graph.

The eigenvectors $S_x^{(m)}$ of the adjacency matrix are labeled by the integer Coxeter exponents m

$$\sum_{x'} A^{xx'} S_{x'}^{(m)} = 2\cos\frac{\pi m}{h} \quad S_x^{(m)}. \tag{3}$$

where h is the dual Coxeter number of the Lie algebra.

In order to simplify the notations we shall always denote the eigenvector with largest eigenvalue (the Perron-Frobenius vector) by S_x,

$$S_x = \begin{cases} S_x^{(1)}, & A, D, E \ ; \\ S_x^{(0)}, & \hat{A}, \hat{D}, \hat{E}. \end{cases} \tag{4}$$

The normalized quantities

$$\chi_x^{(m)} = \frac{S_x^{(m)}}{S_x}. \tag{5}$$

satisfy a closed (fusion) algebra [9]

$$\chi_x^{(m)} \chi_x^{(m')} = \sum_{(m'')} C^0_{mm'm''} \chi_x^{(m'')} \tag{6}$$

and orthogonality conditions of the form

$$\sum_x S_x^2 \ \chi_x^{(m)} \chi_x^{(m')} = \delta_{m,m'}. \tag{7}$$

More generally, we define the genus-g fusion coefficients

$$C^g_{m_1 \cdots m_n} = \sum_x S_x^{2-2g} \ \chi_x^{(m_1)} \cdots \chi_x^{(m_n)}. \tag{8}$$

2.2. Definition of the height models

Let us remind the definition of the height model with target space X on an arbitrary triangulated surface (two-dimensional simplicial complex) $\mathcal{S}$ [5]. Each height configuration on $\mathcal{S}$ represents a map $\mathcal{S} \to X$ such that the heights $x(\sigma), x(\sigma')$ of any two adjacent sites σ, σ' of $\mathcal{S}$ are adjacent or equal in the target space. The partition function of the height model defined on the surface $\mathcal{S}$ is equal to the sum over all height configurations $\mathcal{S} \to X$

$$\mathcal{F}[\mathcal{S}] = \sum_{\mathcal{S} \to X} W(\mathcal{S} \to X). \tag{9}$$

with Dirichlet boundary conditions , i.e., constant height x_i along the connected components $\mathcal{C}_i$ of the boundary $\partial\mathcal{S}$, imposed. The Boltzmann weight of each height configuration is a product of factors associated with the vertices σ and the triangles $\triangle$ of $\mathcal{S}$

$$W(\mathcal{S} \to X) = \prod_{\sigma \in \mathcal{S}} W_\bullet[x(\sigma)] \prod_{\triangle \in \mathcal{S}} W_\triangle[x(\triangle)]. \tag{10}$$

Note that in the case of a regular triangular lattice, where exactly 6 triangles meet at each vertex, the Boltzmann weights associated with the vertices can be distributed among the adjacent triangles. In the case of a lattice with defects we have to take into account the local curvature and the the form (10) of the Boltzmann weights is more appropriate. The local Boltzmann weights are expressed through the components of the Perron-Frobenius vector S_x as follows

$$W_\bullet(x) = S_x \tag{11}$$

$$W_\triangle(x_1, x_2, x_3) = \frac{\delta_{x_1x_2} A_{x_2x_3} A_{x_3x_1}}{\sqrt{S_{x_1}}} + \frac{\delta_{x_2x_3} A_{x_3x_1} A_{x_1x_2}}{\sqrt{S_{x_2}}} + \frac{\delta_{x_3x_1} A_{x_1x_2} A_{x_2x_3}}{\sqrt{S_{x_3}}}. \tag{12}$$

2.3. The height models on a random surface

The height model on a random lattice is the statistical ensemble of all height configurations $\mathcal{S} \to X$ where the surface $\mathcal{S}$ is treated as as an additional dynamical variable. The partition function of the model is defined as the sum over all surfaces $\mathcal{S}$ and all maps $\mathcal{S} \to X$ satisfying certain conditions. Thus the height model on a random lattice can be also viewed as a problem of a random surface $\mathcal{S}$ immersed in the discrete target space X.

Denote by $\Sigma^{g,A}_{l_1,\ldots,l_n}$ the ensemble of all triangulated surfaces $\mathcal{S}$ with genus g, area A, and with boundary $\partial\mathcal{S}$ consisting of n loops $C_1, ..., C_n$ with lengths $l_1, ..., l_n$ For each surface $\mathcal{S}$ we define the partition function $\mathcal{F}[\mathcal{S}, x_1, ..., x_n]$ where x_i denotes the height of loop C_i. The genus g, n-loop amplitude reads

$$\mathcal{F}^{g,A}_{l_1,x_1;\ldots,l_n,x_n} = \sum_{\mathcal{S}} \mathcal{F}(\mathcal{S}; x_1, ..., x_n), \qquad \mathcal{S} \in \Sigma^{g,A}_{l_1,\ldots,l_n}. \tag{13}$$

The partition function of the height model with target space X is by definition the exponent

$$\mathcal{Z}[\kappa, T, g_{kx}] = e^{\mathcal{F}[\kappa, T, g_{kx}]} \tag{14}$$

of the generating function for the loop amplitudes

$$\mathcal{F}[\kappa, T, g_{k,x}] = \sum_{g=0}^{\infty} \kappa^{2g-2+n} \sum_{A \geq 0} \frac{1}{T^A} \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\sum_{l_i>0}\sum_{x_i\in X}\frac{g_{l_1x_1}}{S_{x_1}^{l_1/2}}\dots\frac{g_{l_nx_n}}{S_{x_n}^{l_n/2}}\mathcal{F}^{g,A}_{l_1,x_1;\dots,l_n,x_n}. \quad (15)$$

The mapping of the height models onto a gas of loops yields a powerful combinatorial method allowing to write a complete set of equations for the loop amplitudes. Let us sketch the loop gas method. For given triangulated surface $\mathcal{S}$, each height configuration defines a set of nonintersecting loops on the dual tri-coordinated graph. For example, if $\mathcal{S}$ is the regular triangular lattice, the loops live on the dual honeycomb lattice.

The loops represent the domain walls separating the domains of constant height x. The Boltzmann weight of a loop configuration can be organized as a product of factors associated with its connected domains and the boundaries of the surface [5]. A domain of height x contributes a factor S_x^{2-2g-n}, where n is the number of its boundaries and g is the enclosed genus. With the definition (15) of the generating function, this rule applies also to the domains adjacent to the boundaries. The Boltzmann factors associated with a boundary of length l and height x is g_{lx}.

The fact that the Boltzmann weights of the domains depend only on their topology allows to perform the sum over loops and the sum over surfaces in the opposite order. Namely, we first perform the sum over the shapes of the domains and afterwards the sum over the topologically inequivalent loop configurations. The result of the summation can be expressed as a sum of connected graphs whose vertices and lines label correspondingly the domains and the domain walls. Let $W^{(g)}_{l_1,\dots,l_n}$ be the number of of domains of genus g and n connected boundaries of lengths $l_1,\dots,l_n$ (we assume that each boundary has a marked point). Then the partition function (15) is equal to the sum of all connected Feynman graphs composed from the propagator

$$G_{l,x;l',x'} = A_{x,x'}\ \frac{T^{-l-l'}}{l+l'}\frac{(l+l')!}{l!\ \ l'!}, \quad (16)$$

the vertices

$$V^{(g)}_{x;l_1,\dots,l_n} = W^{(g)}_{l_1,\dots,l_n}\left(\frac{\kappa}{S_x}\right)^{2g-2+n}, (n\geq 2) \quad (17)$$

and the tadpoles

$$V^{(g)}_{x,l} = W^{(g)}_{l_1}\left(\frac{\kappa}{S_x}\right)^{2g-1} + \delta_{g,0}\ g_{lx}. \quad (18)$$

The large l asymptotics of the quantities $W^{(g)}_{l_1,\dots,l_n}$ is given by the loop amplitudes for the so called "topological gravity" whose explicit form is known. The corresponding loop Feynman rules are derived in Section 5.

The loop gas representation is encoded in the following Ward identities for the partition function [5] (14)

$$\Big[-\frac{\partial}{\partial g_{nx}}+\sum_{k\geq 3}kg_{kx}\frac{\partial}{\partial g_{n-2-k,x}}+\sum_{k=0}^{n-2}\frac{\partial}{\partial g_{kx}}\frac{\partial}{\partial g_{n-2-k,x}}+\sum_{x'}A^{xx'}\sum_{k,k'\geq 0}\frac{(k+k')!}{k!\ \ k'!}\frac{1}{T^{k+k'+1}}\frac{\partial}{\partial g_{n-1+k,x}}\frac{\partial}{\partial g_{k'x'}}\Big]\mathcal{Z}=0 \quad (x\in X, n=1,2,\dots) \quad (19)$$

The coefficients of the genus expansion of (19) give identities relating the loop amplitudes of different genus. The leading term in the expansion in κ leads to a closed equation for the disc amplitude $W_{x,l}=\partial\log\mathcal{Z}/\partial g_{k,x}|_{\kappa\to 0}$

$$\Big[Wn,x=\sum_{k\geq 3}kg_{k,x}W_{n-2-k,x}+\sum_{k=0}^{n-2}W_{k,x}W_{n-2-k,x}+\sum_{x'}A^{xx'}\sum_{k,k'\geq 0}\frac{(k+k')!}{k!\ \ k'!}\frac{1}{T^{k+k'+1}}W_{n-1+k,x}W_{k',x'}\Big] \quad (x\in X, n=1,2,\dots). \quad (20)$$

3. ENSEMBLES OF COUPLED RANDOM MATRICES

The crucial observation allowing to establish the equivalence with ensembles of random matrices is that the the vertices (17) can be represented as the loop correlators in a Gaussian $N\times N$ matrix model with $N=S_x/\kappa$

$$\sum_{g=0}^{\infty}N^{2-2g-n}W^{(g)}_{l_1,\dots,l_n} = \int D\mathbf{M}e^{-\frac{N}{2}\mathrm{Tr}\mathbf{M}_x^2}\ \mathrm{Tr}\mathbf{M}^{l_1}\dots\mathrm{Tr}\mathbf{M}^{l_n}. \quad (21)$$

Therefore, if we associate with each height x a hermitian matrix variable $\mathbf{M_x}$ of size $N_x \times N_x$ where

$$N_x = \frac{S_x}{\kappa}, \tag{22}$$

the perturbative expansion of the matrix integral

$$\mathcal{Z}_X[g_{lx}, N_x; x \in X] = \int \prod_x D\mathbf{M}_x$$

$$\exp\Big[\mathrm{Tr}\sum_{x\in X}\Big(-\frac{N_x}{2}\mathbf{M}_x^2 + \sum_{l=1}^{\infty} g_{lx}\mathbf{M}_x^l\Big) + \frac{1}{2}\sum_{x,x'}\sum_{l,l'}\mathrm{Tr}\mathbf{M}_x^l G_{lx;l'x'}\mathrm{Tr}\mathbf{M}_{x'}^{l'}\Big] \tag{23}$$

around the gaussian measure will give the partition function $\mathcal{Z} = e^{\mathcal{F}}$ (15) [12]. One can check that the Ward identities (19) follow from the translational invariance of the integration measure in (23).

The sum in the exponent with the coefficients (16) is a simple logarithm and after shifting the matrix variables as

$$\mathbf{M}_x \to \mathbf{M}_x + T/2 \tag{24}$$

we write the integral (23) as

$$\mathcal{Z}_X[V^x; N_x] = \int \prod_{x\in X} D\mathbf{M}_x e^{-\mathrm{Tr}V^x(\mathbf{M_x})}$$

$$\prod_{x,x'}\Big|\det(1\otimes\mathbf{M}_x + \mathbf{M}_{x'}\otimes 1)\Big|^{-A^{x,x'}/2} \tag{25}$$

where the coefficients in the expansion of the potential

$$V^x(z) = \sum_n T_n^x z^n \tag{26}$$

In particular, the $\hat{A}_0$ model is dual to the $O(2)$ model on a random lattice [3] and its partition function is given by the following $N\times N$ matrix integral

$$\mathcal{Z}_{\hat{A}_0}[V,N] = \int D\mathbf{M}\frac{e^{-\mathrm{Tr}V(\mathbf{M})}}{\Big|\det(1\otimes\mathbf{M} + \mathbf{M}\otimes 1)\Big|}. \tag{27}$$

3.1. Coulomb gas representation

The integrand in (25) depends only on the eigenvalues $\lambda_{ix}, i = 1,2,...,N_x$, of the matrix variable $\mathbf{M_x}$. Therefore we can retain only the radial part of the integration measure $d\mathbf{M}_x \sim \prod_i d\lambda_{ix}\prod_{i<j}(\lambda_{ix}-\lambda_{jx})^2$ and write the partition function (25) as

$$\mathcal{Z}_X[V^x, N_x; x\in X] = \int \prod_{x\in X}\prod_{i=1}^{N_x} d\lambda_{i,x}$$

$$e^{-V^x(\lambda_{i,x})}\prod_{x,x'\in X}\frac{\prod_{i\neq j}(\lambda_{ix}-\lambda_{jx'})^{\delta_{x,x'}}}{\prod_{i,j}|\lambda_{ia}+\lambda_{ja'}|^{A^{xx'}/2}}. \tag{28}$$

Similar integrals have been introduced by the ITEP group as "conformal matrix models" [15]. To make the connection with the construction in [15] let us divide the nodes of the Dynkin diagram X into two groups (say, black and white) so that the nearest neighbors of each node have the opposite color and change the sign of the λ-variables associated with the black points. Denote by ε_x the color (white (+) or black (−)) of the point $x\in X$. Then the integral can be written in the form

$$\mathcal{Z}_X[V^x, N_x; x\in X] = \int \prod_{x\in X}\prod_{i=1}^{N_x} d\lambda_{i,x}$$

$$e^{-V^x(\varepsilon_x\lambda_{ix})}\prod_{(i,x)\neq(j,x')}(\lambda_{ix}-\lambda_{jx'})^{\frac{1}{2}\vec{\alpha}^x\cdot\vec{\alpha}^{x'}} \tag{29}$$

where we used the definition of the Cartan matrix (2). This integral can be interpreted as the partition function of $N = \sum_x N_x$ Coulomb particles with vector charges proportional to the simple roots of the Lie algebra with Dynkin diagram X, restricted on a line and interacting with a common potential.

The integral over the eigenvalues (the positions of the charges) is well defined in the large N limit only if the potential keeps the white charges in the negative half-space $\lambda < 0$ and the black charges in the positive half-space $\lambda > 0$. Then a nonsingular large N limit exists since the charges of the same color do not interact and the the black and white charges that attract each other are separated by a potential wall. The critical situation arises when the edges of the black and white charge densities

meet at the origin. The critical singularity is explained in terms of the original statistical model by the dominance of triangulated surfaces with infinite area.

In what follows we prefer to keep the representation (28) of the Coulomb system, which is more naturally related to our original problem, and which can be considered as a generalization of the partition function of the $\hat{A}_0$ model

$$\mathcal{Z}_{\hat{A}_0}[V,N] = \int \prod_{i=1}^{N} d\lambda_i \; e^{-V(\lambda_i)} \frac{\prod_{i<j}(\lambda_i-\lambda_j)^2}{\prod_{ij}|\lambda_i+\lambda_j|}. \quad (30)$$

The existence of a nonsingular thermodynamical limit then implies that the distribution of the eigenvalues has its support along the negative real axis for all x.

Therefore the integration over the eigenvalues λ_{ix} can be restricted from the very beginning to the interval $\lambda < 0$. This restriction will modify the nonperturbative sector but the genus expansion of the partition function will remain the same.

Note also that, since the point $\lambda = 0$ is outside the support of the spectral density, one can consider a more general potential containing negative powers of λ as well. On the other hand, the algebra simplifies considerably if the even powers in the expansion of the potential $V(\lambda)$ are suppressed.

It is sometimes more advantageous to consider instead of (28) the *canonical* partition function

$$\mathcal{Z}_X[V^x,\mu^x] = \sum_{N_x\geq 0} \prod_x \frac{e^{\mu^x N_x}}{\prod_x N_x!} \; \mathcal{Z}_X[V^x,N_x]. \quad (31)$$

We will assume in the following that the chemical potential μ^x for the charges of type x is absorbed in the potential V^x as a constant term.

3.2. The partition function of the $\hat{A}_r$ model as a Fredholm determinant

Let us introduce the integration kernels

$$K_{x,x'}(\lambda,\lambda') = \frac{e^{-\frac{1}{2}[V^x(\lambda)+V^{x'}(\lambda')]}}{\lambda+\lambda'}$$
$$(x=0,1,...,R; \quad x'=x+1) \quad (32)$$

where the integration goes along the negative real axis. With the help of the Cauchy identity

$$\frac{\prod_{i<j}(\lambda_i-\lambda_j)(\mu_i-\mu_j)}{\prod_{i,j}(\lambda_i+\mu_j)} = \det_{ij} \frac{1}{\lambda_i+\mu_j} \quad (33)$$

we can write the canonical partition function of the $\hat{A}_r$ model (whose target space X is a circle of $R+1$ points) as a Fredholm determinant

$$\mathcal{Z}_{\hat{A}_r} = Det(1+K_{01}K_{12}K_{23}...K_{R0}). \quad (34)$$

In particular, the canonical partition function for the $\hat{A}_0$ model is given by

$$\mathcal{Z}_{\hat{A}_0} = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^{N} d\lambda_i \det_{ij} K(\lambda_i,\lambda_j)$$
$$= \mathrm{Det}(1+K) \quad (35)$$

where

$$K(\lambda,\lambda') = \frac{e^{-\frac{1}{2}[V(\lambda)+V(\lambda')]}}{\lambda+\lambda'}$$
$$V(\lambda) = -\mu - \sum_n t_{2n+1}\lambda^{2n+1}. \quad (36)$$

Fredholm determinants like (35) give the correlation functions in various statistical or QFT models. The simplest case is the two-spin correlation function in the Ising model [16] satisfying the Painlevé III equation. More recently, this Fredholm determinant has been considered in the context of self-avoiding polymers on a cylinder [18].

The representation (34) of the partition functions of the $\hat{A}_r$ models leads to the problem of diagonalization of the kernel (32). This problem has been solved explicitly only in the case of a potential $V(z) = t_1\lambda + t_{-1}\lambda^{-1}$ [17]. An original method to find the eigenfunctions in this case has been suggested by M. Staudacher [20]. Unfortunately the method does not work more complicated potentials. The idea is to look for a differential operator commuting with the integral operator represented by the kernel K and therefore having the same spectrum of eigenstates. If we introduce the parametruzation $\lambda = -e^u$ $(-\infty < u < \infty)$, then the kernel (32) becomes

$$K(u,u') = \frac{e^{-\frac{1}{2}[v(u)+v(u')]}}{\cosh\frac{u-u'}{2}} \quad (37)$$

with potential

$$v(u) \equiv V(e^u) = -\mu - t_1 e^u - t_{-1} e^{-u} \tag{38}$$

and nonrestricted homogeneous measure du. It is easy to check that the linear operator with kernel (37) commutes with the differential operator

$$H = \frac{\partial^2}{\partial u^2} - [\frac{1}{2} v'(u)]^2 \tag{39}$$

and therefore has the same set of eigenstates.

3.3. Loop equations the ADE and $\hat{A}\hat{D}\hat{E}$ matrix models

The resolvent

$$W_x(z) = \lim_{N\to\infty} \langle \sum_{i=1}^{N_x} \frac{1}{z - \lambda_{ix}} \rangle \tag{40}$$

represents a meromorphic function of z with a cut along the support $[a, b]$ $(a < b < 0)$ of the classical eigenvalue density and behaves at infinity as

$$W_x(z) = \frac{N_x}{z} + \mathcal{O}(\frac{1}{z^2}). \tag{41}$$

The loop equations (19) can be derived in the matrix model from the translational invariance of the matrix integration measure. The classical $(\kappa \to 0)$ loop equation for the resolvent $W_x(z)$ reads [6]

$$W_x(z)^2 + \oint_{\mathcal{C}_-} \frac{dw}{2\pi i} \frac{1}{z - w} W_x(w) [-\partial V^x(w) + \sum_{x'} A^{xx'} W_{x'}(-w)] = 0 \tag{42}$$

where the contour $\mathcal{C}_-$ encloses the point z and the cut $[a, b]$ of $W_x(z)$ and leaves outside the cut $[-b, -a]$ of $W_x(-z)$.

The exact loop equations are obtained from (42) by replacing $W_x(z)$ with the loop insertion operator

$$\frac{d}{dV^x(z)} = \sum_{n=1}^{\infty} z^{-n-1} \frac{\partial}{\partial T_n^x} \tag{43}$$

and represent the following quadratic differential constraints on the partition function

$$\oint_{\mathcal{C}_-} \frac{dw}{2\pi i} \frac{1}{z - w} \mathcal{T}^{(x)}(w) \quad \mathcal{Z}_X = 0, \quad x \in X \tag{44}$$

where

$$\mathcal{T}^{(x)}(z) = \frac{d^2}{dV^x(z)^2} + \Big(-\partial V^x(z) + \sum_{x'} A^{xx'} \frac{d}{dV^{x'}(-z)} \Big) \frac{d}{dV^x(z)}. \tag{45}$$

Let us introduce the current-like field

$$\partial \hat{\Phi}_x(z) = \frac{d}{dV^x(z)} - \partial V_x(z). \tag{46}$$

where the covariant components $V_x(z)$ of the potential are defined by

$$V^x(z) = 2V_x(z) - \sum_{x'} A^{xx'} V_{x'}(-z). \tag{47}$$

Then the operators (45) can be written in the form

$$\mathcal{T}^{(x)}(z) = \partial\hat{\Phi}_x(z) [\partial\hat{\Phi}_x(z) + \sum_{x'} A^{xx'} \partial\hat{\Phi}_{x'}(-z)]. \tag{48}$$

The classical expectation value $\partial\Phi_x(z)$ of the current (46) is related to the resolvent (40) by $\partial\Phi_x(z) = W_x(z) - \partial V_x(z)$ and is completely determined by the conditions

$$\Phi_x(\lambda + i0) + \Phi_x(\lambda - i0) - \sum_{x'} A^{xx'} \Phi_{x'}(-z) = 0, \quad (a < \lambda < b); \tag{49}$$

$$\Phi_x(z) = -V_x(z) + N_x \ln z + \mathcal{O}(\frac{1}{z}), (z \to \infty), \tag{50}$$

which are obtained by taking the imaginary part of the classical loop equation (42). The first of these equations means that the meromorphic function

$$\hat{\Phi}^x(z) = 2\hat{\Phi}_x(z) - \sum_{x'} A^{xx'} \hat{\Phi}_{x'}(-z). \tag{51}$$

has vanishing real part along the cut $[a, b]$.

The constraints (44) form a set of mutually commuting Virasoro algebras, one for each point x of the target space. One can interpret the Virasoro constraints for given point x as the loop equations for the topological gravity [21], with potential depending on the fields associated with the adjacent points x'. In this way the solution of the topological gravity appears as the mean field problem in our models. In the next section we formulate a collective field approach utilising the solution of the topological gravity.

4. LOOP DIAGRAM TECHNIQUE

The tree (genus zero) amplitudes can be calculated with the effective collective field theory in which the eigenvalues of the random matrices are considered as a continuous classical liquid. The quantum fluctuations of the liquid interact in a complicated way but the interaction potential can be determined exactly. The strategy is based on the exact solution of the mean field problem, namely the integral over the positions λ_{ix} $(i = 1, ..., N_x)$ of the charges of type x for fixed distribution of the other charges. This leads us to a one-matrix model for generic potential whose solution is known. The free energy of this effective one-matrix model gives the interactions of the vibration modes of the eigenvalue liquid.

Let us introduce collective field $\Phi_x(z)$ and the corresponding lagrange multiplier field $\tilde{V}^x(z)$ by inserting the identity

$$1 = \int \mathcal{D}\Phi\mathcal{D}\tilde{V} \prod_x \exp\Big(\frac{1}{2\pi i}\oint_{\mathcal{C}_-} d[\Phi_x(z) -V_x(z) - \sum_{i=1}^{N_x} \ln(z-\lambda_{ix})]\tilde{V}^x(z)\Big) \quad (52)$$

in the r.h.s. of (28). The integration with respect to the $\lambda_{ix}, i = 1, ...N_x$, yields the mean field free energy $F[\tilde{V}^x]$ defined by

$$e^{F[V]} = \int \prod_{i=1}^{N} d\lambda_i e^{-V(\lambda_i)} \prod_{i<j}(\lambda_i - \lambda_j)^2. \quad (53)$$

After this change of variables the original partition function becomes a functional integral over the collective fields Φ and $\tilde{V}$ with nonrestricted homogeneous measure

$$\mathcal{Z} = \int \mathcal{D}\Phi\mathcal{D}\tilde{V}\ e^{-\mathcal{S}[\Phi_x,\tilde{V}^x;x\in X]}, \quad (54)$$

$$\mathcal{S} = \sum_x \Big[-F[\tilde{V}^x] - \oint \frac{d\Phi_{x'}(z)}{2\pi i}\tilde{V}^x(z)\Big] +\frac{1}{2}\sum_{x,x'} A_{xx'}\oint \frac{d\Phi_{x'}(z')}{2\pi i}\oint \frac{d\Phi_x(z)}{2\pi i}\ln(z+z'). \quad (55)$$

The dependence on the potential $V^x(z)$ is through the asymptotics of the field Φ_x at $z \to \infty$.

The string propagator and vertices are obtained by expanding the effective action (55) around the mean field $\Phi^c_x, \tilde{V}^x_c$ determined by the large N saddle point equations. The dependence on the string coupling constant $\kappa = S_x/N_x$ is through the genus expansion of the effective potential $F[\tilde{V}^x] = \sum_{g\geq 0} N_x^{2-2g}\mathcal{F}^{(g)}[\tilde{V}^x]$. It is convenient to make one more change of variables and introduce the field $\tilde{\Phi}_x(z)$ defined by

$$\partial\tilde{\Phi}_x(z) = -\frac{dF^{(0)}[\tilde{V}^x]}{d\tilde{V}^x(z)} - \frac{1}{2}\partial\tilde{V}^x(z) \quad (56)$$

as an independent fluctuating variable. The fields $\tilde{\Phi}_x(z)$ and $\Phi_x(z)$ have the same classical values $\Phi^c_x(z)$ but on the quantum level they play different roles : the external source is coupled only to the field $\Phi_x(z)$ while the interactions affect only of the field $\tilde{\Phi}_x(z)$.

It is of course possible to consider the discontinuities

$$\phi_x(\lambda) = -\frac{1}{\pi}\Im\Phi_x(\lambda), \quad \tilde{\phi}_x(\lambda) = -\frac{1}{\pi}\Im\tilde{\Phi}_x(\lambda) \quad (57)$$

as independent functional variables. In any case, the functional integration measure in (55) is defined only after introducing a mode expansion for the two fields.

4.1. Saddle point

Upon an appropriate rescaling of the λ-variable, the support $[a, b]$ of the classical spectral density $\rho^c(\lambda) = -\frac{1}{\pi}d\Im\Phi^c_x(\lambda)/d\lambda$ can be taken to be the interval $[-L, -1]$ on the negative real axis.

The solution of the saddle point equation in the continuum limit $L \to \infty$ reads, up to an arbitrary normalization,

$$X = ADE: \qquad \frac{d}{dz}\Phi^c_x(z) = -\frac{S_x}{\kappa}\ \frac{(z+\sqrt{z^2-1})^\beta + (z-\sqrt{z^2-1})^\beta}{2\pi\beta|\sin\pi\beta|} \quad (58)$$

where

$$\beta = 1 \pm \frac{1}{h} + 2m \quad (59)$$

and the string coupling constant scales as $\kappa \sim L^{1+\beta}$. The different branches $(\pm, m)$ correspond the different critical regimes of the height model

(for details see [6] and [14]). The regimes $(-,0)$ and $(+,0)$ are sometimes referred as *dense* and *dilute* phase of the height model. The central charge of the corresponding conformal field theory is

$$C = 1 - 6\frac{(\beta-1)^2}{\beta}. \tag{60}$$

The saddle-point solution for the case $X = \hat{A}\hat{D}\hat{E}$ is obtained as the limit $\beta \to 1$ of (58).

Our aim now is to define the Hilbert space of one-loop states, choose a complete orthonormalized set of eigenstates of the quadratic action, and finally express the interactions in terms of the mode expansion of the collective fields.

4.2. Gaussian fluctuations

It is consistent with the perturbative expansion to assume that the fluctuating fields are again supported by a semi-infinite interval, but its right end can be slightly displaced with respect to its saddle point value -1 due to fluctuations.

Inserting the genus-zero contribution to the mean field free energy (53)

$$F^{(0)}[\tilde{V}_x] = -\frac{1}{\pi}\int \Re\tilde{\Phi}_x d\Im\tilde{\Phi}_x \tag{61}$$

we find for the tree-level action

$$\mathcal{S}^{(0)} = \frac{1}{\pi}\sum_x \int \Big\{\Re\tilde{\Phi}_x(\lambda) d\Im\tilde{\Phi}_x(\lambda)$$
$$+\big[\sum_{x'} A^{xx'}\Phi_{x'}(-\lambda) - 2\Re\tilde{\Phi}_x(\lambda)\big] d\Im\Phi_x(\lambda)\Big\}. \tag{62}$$

The tree-level effective action (62) can be split into gaussian and interacting parts

$$\mathcal{S}^{(0)} = \mathcal{S}^{\text{free}} + \sum_{n\geq 3}\mathcal{S}^{(0,n)} \tag{63}$$

where $\mathcal{S}^{(0,n)}$ is the genus-zero n-loop interaction.

The gaussian fluctuations of the collective fields are those that do not shift the edge of the eigenvalue interval; the fluctuations displacing of the edge are taken into account by the nongaussian terms in the effective action that represent the tree level n-loop interactions.

We therefore define the Hilbert space $\mathcal{H}$ as the space of real functions defined on $X \times [-\infty, -1]$ or, equivalently, the space of analytic fields having a cut along the interval $[-\infty, -1]$.

For the purpose of diagonalizing the quadratic action it is very useful to introduce the map

$$z(\tau) = \cosh\tau \tag{64}$$

transforming the space of meromorphic functions in the z-plane cut along the interval $[-\infty, -1]$ into the space of entire even analytic functions of τ. In the following we will keep the same letters for the analytic fields considered as functions of τ and denote $\Phi(\tau) \equiv \Phi(z(\tau))$. The disc amplitude (58) as a function of τ reads

$$\frac{d}{dz}\Phi^c_x(\tau) = \frac{S_x}{\kappa}\ \frac{\cosh\beta\tau}{2\pi\beta|\sin\pi\beta|}. \tag{65}$$

The operations $\Im$ and $\Re$ in the z-space become finite-difference operators in the τ-space. Denoting $\partial = \partial/\partial\tau$),we can write

$$\Im\Phi(-z) = \sin\pi\partial\Phi(\tau),$$
$$\Re\Phi(-z) = \cos\pi\partial\Phi(\tau) \tag{66}$$

It is clear that the plane waves

$$\langle x, \tau | m, E\rangle = S_x^{(m)} e^{iE\tau} \tag{67}$$

form a complete set of (delta-function) normalized wave functions diagonalizing the quadratic action The latter is given by the expression (62) where the integration is restricted to the interval $\lambda \leq 1$. According to (66), the Fourier components of the fields $\tilde{\phi}$ and ϕ are related to these of $\tilde{\Phi}$ and Φ by $\tilde{\phi}_{(m)}(E) = \frac{1}{\pi}\sinh(\pi E)\tilde{\Phi}_{(m)}(E)$, $\phi_{(m)}(E) = \frac{1}{\pi}\sinh(\pi E)\Phi_{(m)}(E)$. The action (62) reads, in terms of these fields,

$$\mathcal{S}^{\text{free}}[\phi,\tilde{\phi}] = \sum_{p=m/h}\int_0^\infty \frac{dE}{2\pi}\Big[\phi\ \frac{\pi E\cos\pi p}{\sinh\pi E}\phi$$
$$+(\tilde{\phi} - 2\phi)\ \frac{\pi E\cosh\pi E}{\sinh\pi E}\ \tilde{\phi}\Big]. \tag{68}$$

By inverting the quadratic form in (68) we find the propagators in the $(E, p = \frac{m}{h})$ space

$$G^{\phi\phi}(E,p) = G^{\phi\tilde{\phi}}(E,p) = G(E,p),$$
$$G^{\tilde{\phi}\tilde{\phi}}(E,p) = G(E,p) - G(E,\frac{1}{2}) \tag{69}$$

where

$$G(E,p) = \frac{\sinh \pi E}{\pi E} \frac{1}{\cosh \pi E - \cos \pi p} = \frac{2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{E^2 + (p+2n)^2}. \quad (70)$$

The three propagators have the following diagrammatic meaning: $G^{\tilde{\phi}\phi}$ ($G^{\tilde{\phi}\tilde{\phi}}$) is associated with the external (internal) lines of a Feynman diagram, and $G^{\phi\phi}$ is the genus-zero loop-loop correlator.

4.3. Interactions

In order to compute the interaction potential we need the explicit answer for the one-matrix integral.

The free energy $F[V]$ of the one-matrix model (53) is most easily expressed not in terms of the potential $V(z)$, but in terms of the field $\partial\Phi(z) = dF^{(0)}/dV(z) - \frac{1}{2}\partial V(z)$ ([21] - [23]). The meromorphic function $\partial\Phi(z)$ has vanishing real part along its cut $[a = -\infty, b]$ and therefore can be expanded as a series in positive half-integer powers of $z - b$. (Note that $b = b[V]$ depends on the potential!) If z_0 is a point close to b, the field can be nevertheless expanded as a series in the positive and negative half-integer powers of $z - z_0$. The free energy F can be expressed in terms of the coefficients of the nonsingular at $z = z_0$ part of the expansion. The coefficients are given (up to numerical factors) by the linear functionals $(k|\Phi)_{z_0}, k = 0, 1, ...$ defined by the generating function

$$\sum_{k=0}^{\infty} \frac{u^k}{k!} (k|\Phi)_{z_0} = \sqrt{2} \oint \frac{dz}{2\pi i} \frac{d\Phi^c(z)/dz}{\sqrt{z - z_0 - 2u}}. \quad (71)$$

Sometimes they are called KdV coordinates of the field Φ. The KdV coordinates at different expansion points are related by

$$(\Phi|n)_{z_0+2u} = \sum_{k\geq 0} (\Phi|n+k)_{z_0} \frac{u^k}{k!}. \quad (72)$$

The choice of the expansion point z_0 is a matter of convenience. The most compact expression corresponds to the choice $z_0 = b$, where b is the position of the branchpoint of $\Phi(z)$, imposed by the condition

$$(\Phi|0)_b = 0. \quad (73)$$

In this case the explicit expression for of $F[\Phi]$ reads [22]

$$F[\Phi] = F^{(0)}[\Phi] - \frac{1}{24} \ln(1|\Phi)_b + \sum_{2g-2+n>0} (1|\Phi)_b^{2-2g} \frac{(-1)^n}{n!} \sum_{\substack{k_1,\ldots,k_n \geq 2 \\ k_1+\cdots+k_n = 3g-3+n}} \{k_1 \cdots k_n\}_g \frac{(k_1|\Phi)_b}{(1|\Phi)_b} \cdots \frac{(k_n|\Phi)_b}{(1|\Phi)_b}. \quad (74)$$

The coefficients $\{k_1 \cdots k_n\}_g$ $(k_1 + \cdots + k_n = 3g-3+n)$ are the the intersection numbers on the moduli space $\mathcal{M}_{g,n}$ of algebraic curves of genus g with n marked points [23]. The intersection numbers can be obtained from a system of recurrence relations equivalent to the loop equations [21]. In particular, the genus-zero the intersection numbers coincide with the multinomial coefficients

$$\{k_1 \cdots k_n\}_0 = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}, \quad k_1 + \cdots + k_n = n - 3. \quad (75)$$

The genus g term in the expansion of the free energy is given by the restricted sum (74) with $k_1 + \cdots + k_n = 3g - 3 + n$, which contains only finite number of terms.

The expression of the free energy as a function of the KdV coordinates $(n|\Phi)_{z_0}$ with fixed z_0 has the same form but the sum over $k_1, ..., k_n$ should be taken in the range $k_i \geq 0$ since the coordinate $(0|\Phi)_{z_0}$ no longer vanishes. Note that F is singular at $\Phi = 0$ and has to be expanded around some nontrivial background $\Phi^c(z)$. The simplest nontrivial background to expand about is the "topological" background

$$\Phi^c(z) = \frac{1}{\sqrt{2}} \frac{2}{3} \frac{(z - z_0)^{3/2}}{\kappa} \quad (76)$$

with only one nonvanishing coordinate $(\Phi^c|1)_{z_0} = 1/\kappa$. A generic potential can be considered as a deformation about this topological point. For deformation parameters $t_k = t_k(\Phi)$ defined as

$$(k|\Phi)_{z_0} = \frac{1}{\kappa}(\delta_{k,1} - t_k) \quad (77)$$

the genus-g term of the free energy reads, in terms

of t_k,

$$F^{(g)}[\Phi] = \kappa^{2g-2+n} \sum_{n\geq 0} \frac{1}{n!} \sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n=3g-3+n}} \{k_1 \cdots k_n\}_g \; t_{k_1} \cdots t_{k_n}. \quad (78)$$

Now we are ready to find the vertices (17) for the genus g, n-loop interactions $(2g+n-2>0)$. We have to expand the effective potential $F[\tilde{\Phi}_x]$ around the background (58) and taking $z_0=-1$. Alternatively, we can expand around the topological background (76) with $z_0=-1$ and consider the background (58) as a perturbation. We prefer to do the latter because in this case the interaction vertices do not depend on the background. Then the deformation parameters $t_{kx} = t_k(\tilde{\phi}_x)$ are defined by

$$\delta_{k,1} - t_{kx} = \frac{(k|\tilde{\Phi}_x)}{(1|\Phi^c_x)}. \quad (79)$$

(Once the expansion point is $z_0=-1$ is fixed, we will write $(k|*)$ instead of $(k|*)_{-1}$.) We know the explicit form of the propagator and the tadpole in the base of plane waves, while the interaction is expressed in terms of the KdV coordinates. Therefore we need the transformation matrix between these two mode expansions. The generating function for the KdV coordinates is given, in the τ space, by

$$\sum_{k=0}^{\infty} \frac{u^k}{k!}(k|\tilde{\Phi}) = \int_0^\infty d\tau \frac{\partial_\tau \cos\pi\partial_\tau \tilde{\phi}(\tau)}{\sqrt{\cosh^2\frac{\tau}{2}-u}}. \quad (80)$$

For $\tilde{\phi}(\tau) = \sin E\tau$ the integral is a Legendre function and its expansion in u gives for the KdV coordinates of a plane wave

$$(k|E) = \ \pi E\ \Pi_k(iE),\ k=0,1,2,... \quad (81)$$

where

$$\Pi_0(a) = 1;$$
$$\Pi_k(a) = \frac{1}{k!}\prod_{j=0}^{k-1}[(j+\frac{1}{2})^2 - a^2],\ k=1,2,... \quad (82)$$

Eq. (81) means that the linear functional $(k|$ acts in the space of odd functions $\tilde{\phi}(\tau) = -\tilde{\phi}(-\tau)$ smooth at $\tau=0$, as

$$(k|\tilde{\Phi}) = \pi[\Pi_k(\partial_\tau)\ \partial_\tau\tilde{\phi}(\tau)]_{\tau=0}. \quad (83)$$

The KdV coordinates of the string background (58) are obtained using the identity $(k|d\tilde{\phi}/dz) = \frac{1}{2}\ (k+1|\tilde{\phi})$

$$(k|\tilde{\phi}^c_x) = -\frac{S_x}{\kappa}\ \Pi_{k-1}(\beta),\ k=2,3,.... \quad (84)$$

The interaction vertices for the field $\tilde{\Phi}_x$ are obtained by expanding the free energy of the one-matrix model around the deformation parameters t^c_k characterizing the saddle-point solution. From (84) we find

$$t^c_0 = t^c_1 = 0,\quad t^c_k = -\Pi_{k-1}(\beta),\ k=2,3,...) \quad (85)$$

After going to the momentum space the factor S_x^{2g-2+n} in the genus expansion of $F[\Phi_x]$ yields the genus g fusion coefficient (8) and the vertex factorises into a product of an intersection number and a fusion coefficient. The KdV representation of the internal propagator is given by the symmetric matrix

$$G_{kk'}(p) = (k|(k'|G_{\tilde{\phi}\tilde{\phi}}(p))); \quad (86)$$
$$G_{00}(p) = -\Pi_1(1-p),$$
$$G_{01}(p) = G_{10}(p) = -\Pi_2(1-p),$$
$$G_{11}(p) = -2\Pi_2(1-p) - 2\Pi_3(1-p),... \quad (87)$$

Finally, the external line factors read

$$(k,p|G_{\tilde{\phi}\phi}|\tau,p\rangle = \Pi_k(\partial_\tau)\frac{\sinh(1-p)\tau}{\sin\pi p\sinh\tau}. \quad (88)$$

To summarize, we have found the following loop Feynman rules:

Propagator :

$$G_{kk'}(p),\quad p = \frac{m}{h} \quad (89)$$

Tadpole :

$$V^{(0)}_{k,m} = \frac{1}{\kappa}t^c_k\delta_{m,1} \quad (90)$$

Vertices : $\quad (g+2n\geq 2)$

$$V^{(g)}_{k_1,m_1,\ldots,k_n,m_n}$$
$$= \kappa^{2g-2+n}\{k_1\cdots k_n\}_g\ C^{(g)}_{m_1\ldots m_n}. \quad (91)$$

Let us give several examples.

(i) The three-loop genus-zero amplitude

$$\langle \Phi_{m_1}(z_1)\Phi_{m_2}(z_2)\Phi_{m_3}(z_3)\rangle_0 = \kappa C^0_{m_1m_2m_3}\prod_{s=1}^{3}\frac{\sinh(\frac{h-m_s}{h})\tau_s}{\sin\pi\frac{m_s}{h}\sinh\tau_s}. \qquad (92)$$

(ii) The genus-one one-loop amplitude

$$\langle\Phi_m(z)\rangle_1 = \kappa\Big[C^1_m\{1\}_1\Pi_1(\partial_\tau) + C^1_m\{02\}_1 t^c_2 + \frac{1}{2}\sum_{m'} C^0_{mm'm'}\{000\}_0 G_{00}(m')\Big]\frac{\sinh(\frac{h-m}{m})\tau}{\sin\pi\frac{m}{h}\sinh\tau}$$
$$= \kappa\delta_{m,1}\sum_{m'}\Big[\frac{1}{24}\Big(\beta^2-\frac{\partial^2}{\partial\tau^2}\Big)+\frac{1}{2}\Big(\frac{m'}{h}-\frac{1}{2}\Big)\Big(\frac{m'}{h}-\frac{3}{2}\Big)\Big]\frac{\sinh(\frac{h-1}{h})\tau}{\sin\frac{\pi}{h}\sinh\tau}. \qquad (93)$$

(iii) The four-loop genus-zero amplitude

$$\langle\prod_{s=1}^{4}\Phi_{m_s}(z_s)\rangle_0 = \kappa^2\Bigg[\Big[\beta^2-\frac{1}{4}+\sum_{s=1}^{4}\Big(\frac{1}{4}-\frac{\partial^2}{\partial\tau_s^2}\Big)\Big]C^0_{m_1m_2m_3m_4} + \sum_m\Big(C^0_{m_1m_2m}C^0_{mm_3m_4} + C^0_{m_1m_3m}C^0_{mm_2m_4} + C^0_{m_1m_4m}C^0_{mm_2m_3}\Big)\Big(\frac{m}{h}-\frac{1}{2}\Big)\Big(\frac{m}{h}-\frac{3}{2}\Big)\Bigg]\prod_{s=1}^{4}\frac{\sinh(1-\frac{m_s}{h})\tau_s}{\sin\pi\frac{m_s}{h}\sinh\tau_s}. \qquad (94)$$

The expression for the genus 1 tadpole (93) is in accord with the recent calculation by B. Eynard and C. Kristjansen in the $O(n)$ model. (In this case one has to put $\pi m/h = \arccos(n/2)$.) Eq. (94) reproduces in the limit $R\to\infty$ the four-loop amplitude found in ref. [24].

5. VERTEX OPERATOR CONSTRUCTION

The relation between the Coulomb gas systems and the integrable hierarchies of differential soliton equations gives another powerful approach to study these systems, especially in application to nonperturbative phenomena.

The relevance of the integrable hierarchies to the critical phenomena on random surfaces is now an established fact. It has been shown by M. Douglas [25] that the chains of random matrices with nearest neighbor interaction ($\mathrm{Tr}\mathbf{M_x M_{x+1}}$) are described by the A series of generalized KdV flows in the classification of Drinfeld and Sokolov [26]. Moreover, as it was noticed by Di Francesco and Kutasov [27], a particular case of the D series reproduces the known qualitative features of the D models of matter fields coupled to gravity .

It remains an open problem, however, how to construct the hierarchies associated with the rational conformal theories coupled to gravity in a unified fashion, including the exceptional ones. We believe that the ADE and $\hat{A}\hat{D}\hat{E}$ matrix models can be used as a starting point to solve this problem. The logic is the following. Each matrix model partition function in the form of Coulomb gas defines a bosonic vertex operator construction. Then the associated hierarchy of soliton equations can be written as a system of Hirota bilinear equations [30] following the general prescription proposed in [29]. In the case $\hat{A}_0$ this is KdV hierarchy (just as in the case of the A_1 model, but of course with different boundary conditions).

Below we show how to make the first step, namely the bosonic vertex operator construction. We restrict ourselves to the A and $\hat{A}$ series, but the generalization to the other cases is evident. Then we give the fermionic representation for the A and $\hat{A}$ series. The boson-fermion correspondence allows to establish a fermionic representation for the $A, \hat{A}, D$ and $\hat{D}$ series, but not for the exceptional (E and $\hat{E}$) cases. There should be certainly a connection between the hierarchies associated with the A and D Coulomb systems and the Lax representations by Drinfeld and Sokolov for the A and D series [26].

5.1. The case $\hat{A}_0$

Before presenting the general construction let us consider in detail the simplest nontrivial example, namely the $\hat{A}_0$ model whose the target space is a circle with one point.

Let us first note that if the integration with respect to λ is replaced by contour integration with respect to the complex variables z_i, the partition function will still satisfy the same loop equations

(44).

We start with the construction of the bosonic field representation of the canonical partition function

$$\mathcal{Z}_{\hat{A}_0} = \sum_{N=0}^{\infty} \int \prod_{i=1}^{N} dz_i \; e^{-V(z_i)} \frac{\prod_{i<j}(z_i - z_j)^2}{\prod_{ij}(-z_i - z_j)}. \quad (95)$$

Introduce the bosonic field $\varphi(z)$ with mode expansion

$$\varphi(z) = \hat{q} + \hat{p}\ln z + \sum_{n\neq 0} \frac{J_n}{n} z^{-n}, \quad (96)$$

$$[J_n, J_m] = n\delta_{m+n,0}; \quad [\hat{p}, \hat{q}] = 1. \quad (97)$$

and the vacuum state $|l\rangle$ defined by

$$J_n|l\rangle = 0, \quad (n > 0); \quad \hat{p}|l\rangle = l|l\rangle. \quad (98)$$

The associated normal ordering is defined by putting $J_n, n > 0$ to the right. Following ref. [31] we define the antisymmetric field

$$\Phi(z) = \varphi(z) - \varphi(-z). \quad (99)$$

The field $\Phi(z)$ and the vertex operator $:e^{\Phi(z)}:$ satisfy the OPE

$$\Phi(z)\Phi(z') =: \Phi(z)\Phi(z') : \\ +2\ln(z - z') - 2\ln(z + z'), \quad (100)$$

$$: e^{\Phi(z)} :: e^{\Phi(z')} := \frac{(z - z')^2}{(z + z')^2} : e^{\Phi(z)} e^{\Phi(z')} : . \quad (101)$$

Define the Hamiltonian

$$H[t] = \sum_n t_n J_n, \quad (102)$$

and the operator

$$G_{A_0} = \exp\left(e^{\mu} \oint \frac{dz}{2\pi i} : e^{\Phi(z)} :\right). \quad (103)$$

It is easy to see that the vacuum expectation value

$$\tau_0[t] = \langle 0|e^{H[t]} G_{A_0}|0\rangle \quad (104)$$

is equal to the partition function (95) with potential

$$V(z) = -\mu - \sum_{n\geq 0} t_{2n+1} z^{2n+1}. \quad (105)$$

The fermionic representation of the partition function of the $\hat{A}_0$ model follows from the bosonization formulas

$$\psi(z) =: e^{-\varphi(z)} : , \psi^*(z) =: e^{\varphi(z)} : \\ \partial\varphi(z) =: \psi^*(z)\psi(z) : \quad (106)$$

where the fermion operators

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r z^{-r-\frac{1}{2}}$$

$$\psi^*(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi^*_{-r} z^{-r-\frac{1}{2}} \quad (107)$$

satisfy the modes in the expansion of the anticommutation relations

$$[\psi_r, \psi^*_s]_+ = \delta_{rs}. \quad (108)$$

The operators (102) and (103) are represented by

$$H[t] = \sum_{n>0} t_n \sum_r : \psi^*_{r-n}\psi_r : \quad (n \in \mathbb{Z}) \quad (109)$$

$$G = \exp\left[e^{\mu} \oint \frac{dz}{2\pi i} : \psi(z)\psi^*(-z) :\right]. \quad (110)$$

and the vacuum states with given electric charge l satisfy

$$\langle l|\psi_{-r} = \langle l|\psi^*_r = 0 \quad (r > l) \quad (111)$$

$$\psi_r|l\rangle = \psi^*_{-r}|l\rangle = 0 \quad (r > l). \quad (112)$$

Following the line of arguments of ref. [28], we can consider the canonical partition function (95) as the large N limit of N-soliton solutions of the KdV hierarchy of integrable differential equations (where t_1 is the space variable and $t_3, t_5, ..$ the time variables). Therefore the partition function (95) represents itself a τ-function of the KdV hierarchy.

More generally, one can define

$$\tau_l[t] = \langle l|e^{H[t]} G_{A_0}|l\rangle. \quad (113)$$

for any integer l but in fact there are only two different τ-functions corresponding to $l = 0(mod\ 2)$ and $l = 1(mod\ 2)$. In terms of Fredholm determinants

$$\tau_l = \det(1 + (-)^l K) \quad (114)$$

where the kernel K is defined by (36).

The KdV and mKdV hierarchies of differential equations are generated by the Hirota bilinear equations [30]

$$\oint \frac{dz}{2\pi i} z^{l-l'} \exp\Big(\sum_n (t_n - t'_n) z^n\Big)$$
$$\tau_l(t_n - \frac{1}{n} z^{-n})\ \tau_{l'}(t'_n + \frac{1}{n} z^{-n}) = 0. \qquad (115)$$

Namely, the second derivative

$$u = 2\frac{\partial^2}{\partial t_1^2} \log \tau_0 \qquad (116)$$

satisfies the KdV hierarchy of differential equations, the first of which is the classical KdV equation

$$\frac{\partial u}{\partial t_3} = 6u\frac{\partial u}{\partial t_1} + \frac{\partial^3 u}{\partial t_1^3}. \qquad (117)$$

Further, the function

$$v = \frac{\partial}{\partial t_1} \log \frac{\tau_0}{\tau_1} \qquad (118)$$

satisfies the modified KdV equation

$$\frac{\partial v}{\partial t_3} = -6v^2\frac{\partial v}{\partial t_1} + \frac{\partial^3 v}{\partial t_1^3} \qquad (119)$$

and u and v are related by the Miura transformation

$$u = -v^2 - \frac{\partial v}{t_1}. \qquad (120)$$

Finally, the bosonic and fermionic representations generalize trivially if we add negative odd powers to the potential,

$$\tau_l[t] = \langle l| e^{\sum_{n\geq 0} t_n z^n}\ G_{A_0}\ e^{\sum_{n<0} t_n z^n} |l\rangle, \qquad (121)$$

and the τ-functions (121) solve the affine sinh-Gordon hierarchy (with $t_{-1}, t_{-3}, ...$ as time variables). The lowest order equations are

$$\frac{\partial^2 \phi}{\partial t_1 \partial t_{-1}} = \frac{1}{2}\sinh 2\phi, \qquad (122)$$

$$-\frac{\partial^2}{\partial t_1 \partial t_{-1}} \log \tau_1 = \frac{e^{2\phi} - 1}{4} \qquad (123)$$
$$(\phi = \log \frac{\tau_0}{\tau_1}).$$

The relation between the Fredholm determinants (114) and the mKdV and sine-Gordon hierarchy has been established directly in [19].

5.2. Bosonic construction in the general case

The idea is to construct the operators $\Phi^x(z)$, $x \in X$, associated with the roots of the corresponding classical Lie algebra, with OPE

$$\Phi^x(z)\Phi^{x'}(z') =: \Phi^x(z)\Phi^{x'}(z') :$$
$$+2\delta_{xx'} \ln(z - z') - A^{xx'} \ln(z + z'). \qquad (124)$$

We will restrict ourselves to the cases A_r and $\hat{A}_r$; the other simply laced algebras can be considered in a similar way.

We will use the fact that the simple roots $\vec{\alpha}^x$ $(x = 1, ..., r)$ and the lowest root $\vec{\alpha}^0$ of A_r can be represented as differences of orthonormal vectors $\vec{e}_1, ..., \vec{e}_{r+1}$ in $\mathbb{R}^{r+1}$

$$\vec{\alpha}^x = \vec{e}_x - \vec{e}_{x+1}, \quad (x = 0, ..., r)$$
$$\vec{\alpha}^0 = \vec{e}_{r+1} - \vec{e}_1. \qquad (125)$$

With each vector $\vec{e}_a$ $(a = 1, 2, ..., r + 1)$ we associate the bosonic field $\varphi_a(z)$ having mode expansion

$$\varphi_a(z) = \hat{q}_a + \hat{p}_a \ln z + \sum_{n\neq 0} \frac{J_{an}}{n} z^{-n}, \qquad (126)$$

with

$$[J_{an}, J_{a'm}] = n\delta_{a,a'}\delta_{n+m,0}, \qquad (127)$$

$$[\hat{p}_a, \hat{q}_{a'}] = \delta_{a,a'}. \qquad (128)$$

The bosonic Fock space $\mathcal{F}_{\vec{l}}$, $\vec{l} \equiv \{l_1, ..., l_{r+1}\}$ is generated by the action of the negative modes on the vacuum state $|\vec{l}\rangle = e^{\sum l_a \hat{q}_a}|\vec{0}\rangle$ satisfying

$$\hat{p}_a|\vec{l}\rangle = l_a|\vec{l}\rangle; \quad J_{an}|\vec{l}\rangle = 0,\ n > 0. \qquad (129)$$

The state $\langle\vec{l}|$ is defined similarly, with the normalization $\langle\vec{l}|\vec{l'}\rangle = \prod_a \delta_{l_a, l'_a}$.

The fields

$$\Phi^x(z) = \varphi_x(z) - \varphi_{x+1}(-z) \quad (x = 1, ..., r)$$
$$\Phi^0(z) = \varphi_{r+1}(z) - \varphi_1(-z) \qquad (130)$$

satisfy (124) and therefore the corresponding vertex operators have OPE

$$: e^{\Phi^x(z)} :: e^{\Phi^{x'}(z')} := \frac{(z - z')^{2\delta_{xx'}}}{(z + z')^{\delta_{x,x'+1} + \delta_{x,x'-1}}}$$
$$: e^{\Phi^x(z)} e^{\Phi^x(z')} : . \qquad (131)$$

Define the Hamiltonian

$$H[\vec{t}\,] = \sum_{a=1}^{r+1}\sum_{n=0}^{\infty} t_{an}J_{an}. \tag{132}$$

and the operator creating the electric charges with fugacity e^{μ}

$$G_{\hat{A}_r} = \exp\left(e^{\mu}\sum_{x=0}^{r} : \oint \frac{dz}{2\pi i} e^{\Phi^x(z)} : \right). \tag{133}$$

Then the vacuum expectation value

$$\tau_{\vec{0}}[\vec{t},\mu] = \langle\vec{0}|e^{H[\vec{t}\,]}G|\vec{0}\rangle \tag{134}$$

is equal to the canonical partition function (28) for the model $\hat{A}_r$ with potential

$$V^x(z) = -\mu - \sum_n z^n[t_{xn} - (-)^n t_{x+1,n}]. \tag{135}$$

More generally, one can define the τ-functions associated with vacuum states with different vector charges $\vec{l}$ and $\vec{l'}$ such that $l_1 + ... + l_{r+1} = l'_1 + ... + l'_{r+1}$

$$\tau_{\vec{l},\vec{l'}}[\vec{t}\,] = \langle\vec{l}|e^{H[\vec{t}\,]}G_{\hat{A}_r}|\vec{l'}\rangle. \tag{136}$$

The bosonic representation of the A_r model is constructed in the same way. The canonical partition function is given by

$$\mathcal{Z}_{A_r}[\vec{t}] = \langle\vec{0}|e^{H[\vec{t}\,]}G_{A_r}|\vec{0}\rangle \tag{137}$$

where

$$G_{A_r} = \exp\left(\sum_{x=1}^{r}\oint \frac{dz}{2\pi i} : e^{\Phi^x(z)} : \right) \tag{138}$$

and

$$t_{x0} - t_{x+1,0} \equiv \mu^x = \frac{S^x}{\kappa}$$

$$S^x = 2S_x - A_{xx'}S_{x'} = 4\sin^2\left(\frac{\pi}{2h}\right) S_x. \tag{139}$$

5.3. Fermionic representation of the $\hat{A}_r$ and A_r models with $r \geq 2$

The fermionic representation of the partition functions follows from the bosonization formulas

$$\psi_a(z) =: e^{-\varphi_a(z)} :, \tag{140}$$

$$\psi^*_a(z) =: e^{\varphi_a(z)} :, \tag{141}$$

$$\partial\varphi_a(z) =: \psi^*_a(z)\psi_a(z) : . \tag{142}$$

Let us introduce the fermion operators $\psi_{r,a}, \psi^*_{r,a}$ ($r \in \mathbb{Z} + \frac{1}{2}, a = 1, ..., R+1$) satisfying the anticommutation relations

$$[\psi_{r,a}, \psi^*_{s,b}]_+ = \delta_{rs}\delta_{ab}. \tag{143}$$

Define the zero-particle states $|-\infty\rangle, \langle-\infty|$ by

$$\psi_{r,a}|-\infty\rangle = 0, \langle-\infty|\psi^*_{r,a} = 0 \tag{144}$$

and the states $|\vec{l}\rangle = |l_1, ..., l_{r+1}\rangle$, $\langle\vec{l}| = \langle l_1, ..., l_{r+1}|$, $(l_a \in \mathbb{Z})$ by

$$|\vec{l}\rangle = \prod_a \prod_{r<l_a} \psi^*_{r,a}|-\infty\rangle$$

$$\langle\vec{l}| = \langle-\infty|\prod_a \prod_{r<l_a} \psi_{r,a}. \tag{145}$$

These states satisfy

$$\langle\vec{l}|\psi_{-r,a} = \langle\vec{l}|\psi^*_{r,a} = 0 \quad (r > l_a) \tag{146}$$

$$\psi_{r,a}|\vec{l}\rangle = \psi^*_{-r,a}|\vec{l}\rangle = 0 \quad (r > l_a). \tag{147}$$

Define the current operators $J_{n,a}$

$$J_{n,a} = \sum_r : \psi^*_{r-n,a}\psi_{r,a} : \quad (n \in \mathbb{Z}) \tag{148}$$

where

$$: \psi^*_{r,a}\psi_{s,b} := \psi^*_{r,a}\psi_{sb} - \langle 0|\psi^*_{r,a}\psi_{sb}|0\rangle, \tag{149}$$

or, equivalently,

$$J_a(z) = \sum_{n\in\mathbb{Z}} J_{n,a}\ z^{-n-1} =: \psi^*_a(z)\psi_a(z) : \tag{150}$$

where

$$\psi_a(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_{r,a}z^{-r-\frac{1}{2}},$$

$$\psi^*_a(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi^*_{-r,a}z^{-r-\frac{1}{2}}. \tag{151}$$

Introduce the Hamiltonian H by (132) such that

$$e^{H(t)}\psi_a(z)e^{-H(t)} = e^{\sum_{n=1}^{\infty} t_{n,a}z^n}\psi_a(z)$$

$$e^{H(t)}\psi^*_a(z)e^{-H(t)} = e^{-\sum_{n=1}^{\infty} t_{n,a}z^n}\psi^*_a(z). \tag{152}$$

Then the operators (133) and (138) in the definition of the τ functions (134) and (137) are represented by

$$G_{\hat{A}_r} = \exp e^{\mu}\oint\frac{dz}{2\pi i}\Big[\sum_{x=1}^{r} : \psi_x(z)\psi^*_{x+1}(-z) : + : \psi_{r+1}(z)\psi^*_1(-z) : \Big] \tag{153}$$

and

$$G_{A_r} = \exp\Big[\oint \frac{dz}{2\pi i}\sum_{x=1}^{r} :\psi_x(z)\psi^*_{x+1}(-z):\Big]. \qquad (154)$$

The τ-functions for the A_r and $\hat{A}_r$ models satisfy the Hirota bilinear equations [30] following from the commutatativity of the operators $S = \sum_a \oint dz\psi_a(z) \otimes \psi^*_a(z)$ and $G \otimes G$. Bilinear equations for the D and E series can be written as well but they have more complicated form [29].

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